Sheffe's Test For Steak Packaging Data

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Creation of the data frame

We define a dataset which contains the results of the following experiment: 12 steaks are randomly divided into 4 even treatment groups, and they are packaged under various conditions (commercial wrap only, vacuum, mixed gas atmosphere, and CO_2 atmosphere). The measured quantity is the logarithm of the number of bacteria on the surface of the steak.

```
> Treatment <- rep(c("Commercial", "Vacuum", "Mixed.Gas", "CO2"),
  each=3)
```

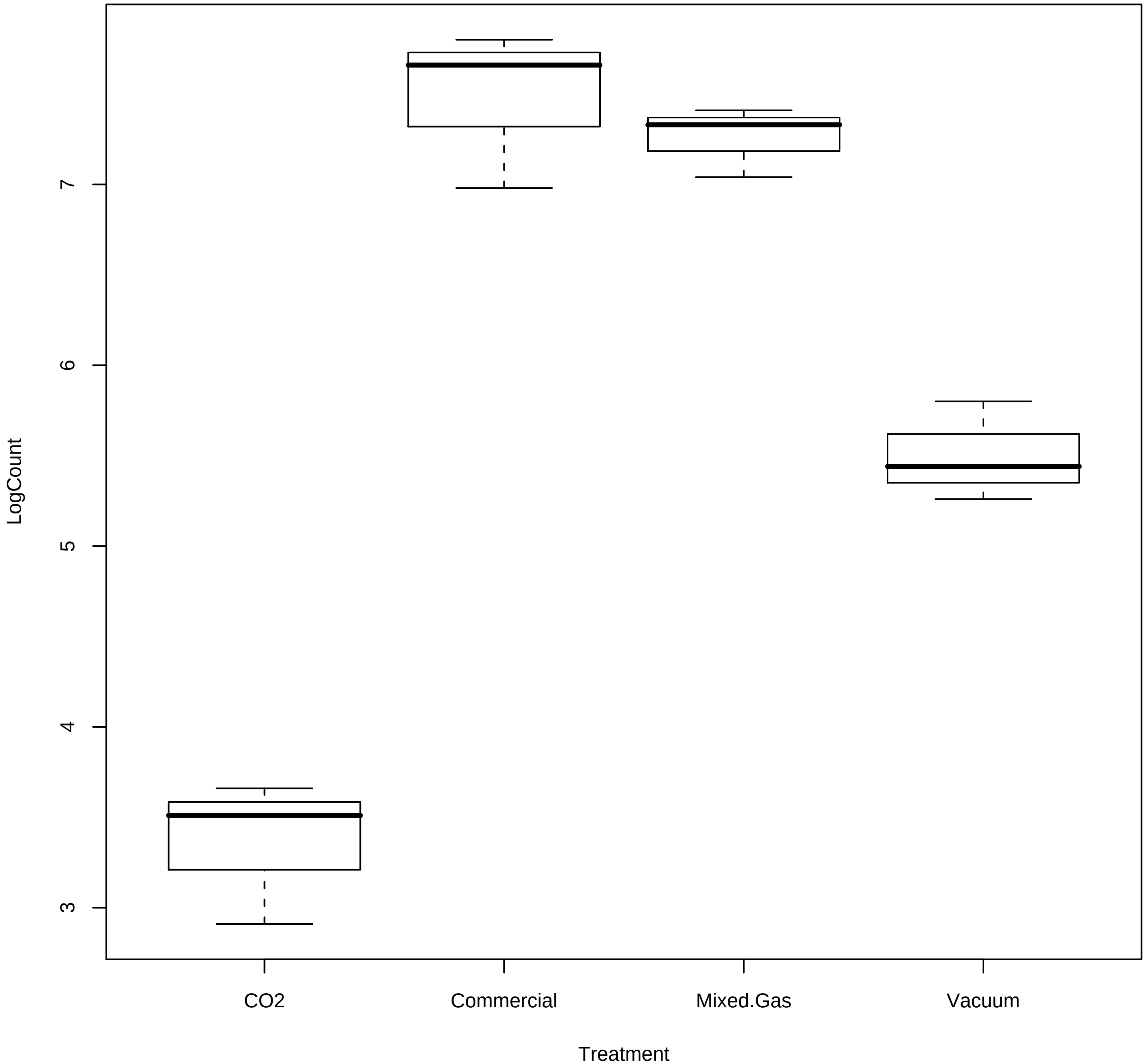
```
> LogCount <- c(7.66, 6.98, 7.80, 5.26, 5.44, 5.80, 7.41, 7.33,
  7.04, 3.51, 2.91, 3.66)
```

```
> steak.data <- data.frame(Treatment, LogCount)
```

```
> steak.data
```

```
    Treatment LogCount
1  Commercial     7.66
2  Commercial     6.98
3  Commercial     7.80
4      Vacuum     5.26
5      Vacuum     5.44
6      Vacuum     5.80
7   Mixed.Gas     7.41
8   Mixed.Gas     7.33
9   Mixed.Gas     7.04
10        CO2     3.51
11        CO2     2.91
12        CO2     3.66
```

```
> X11(pointsize = 6); opts <- options(); opts$texmacs$height <-
  7.5; opts$texmacs$width=7.5; opts$texmacs$nox11=F;
  options(opts);plot(steak.data); v()
```



LogCount
3
4
5
6
7
CO2
Commercial
Mixed.Gas
Vacuum
Treatment

```
>
```

Picking of the contrast

We will test the hypothesis that CO_2 is better then the average of all other treatments. We note that for Sheffe's Test it is not necessary that the contrast is a planned contrast, i.e. that it has been devised in advance of the experiment to support a particular research hypothesis. We could perform the experiment, and then test the conjecture that the CO_2 treatment is better. The test provides for testing such ad hoc research hypotheses.

```
> steak.contrast <- c(-3, 1, 1, 1)
```

```
>
```

We note that with Sheffe's test we do not test the one-directional hypothesis, but the bi-directional, i.e. we test whether the contrast is zero or non-zero. A small modification is needed to test the sign of the contrast.

Performing Sheffe's test

Let C be an unplanned contrast. Thus, C is a linear combination of the means:

$$C = \sum_{i=1}^{t} k_i \mu_i$$

Let $\hat{C}$ be the estimator of the contrast C:

$$\hat{C} = \sum_{i=1}^{t} k_i \hat{\mu_i}$$

where $\mu_i = \sum_{j=1}^{r_i} X_{ij}$ is the group mean. Sheffe's test may be performed by first calculating the Studentized value of the contrast estimator $\hat{C}$:

$$U = \frac{\hat{C}}{\hat{s}\sqrt{\sum_{i=1}^{t} \frac{k_i^2}{r_i}}} = \frac{\hat{C}}{\hat{s}\,\|C\|}$$

We note that the denominator is the square root of the variance estimator, as

$$\mathrm{Var}(\hat{C}) = \sum_{i=1}^{t} k_i^2 \,\mathrm{Var}(\hat{\mu}_i) = \sum_{i=1}^{t} k_i^2 \frac{\sigma^2}{r_i} = \sigma^2 \|C\|^2$$

where $\|C\|$ is the norm of the contrast with respect to the inner product

$$\langle C, D\rangle = \sum_{i=1}^{t} \frac{k_i \cdot d_i}{r_i}$$

where $D = \sum_{i=1}^{t} d_i \mu_i$ is another arbitrary contrast.
Sheffe's theorem says that

$$\mathbb{P}(U \geq U_\alpha) \leq \alpha$$

if

$$U_\alpha = \sqrt{(t-1)F_{\alpha,t-1,N-t}}$$

where F_{α,ν_1,ν_2} is the value of the Fischer F-distribution with degrees of freedom ν_1 and ν_2 for argument $1-\alpha$. We should note that Sheffe's statistic (defined by the last equation) is an approximation, and thus it is generally not true that $\mathbb{P}(U \geq U_\alpha) = \alpha$.

Calculating groups and group means

Our data is given as a data frame, but it is easy to divide it into treatment groups and compute the groups means:

```
> (steak.groups <- split(steak.data$LogCount,
  steak.data$Treatment))
```

```
$CO2
[1] 3.51 2.91 3.66

$Commercial
[1] 7.66 6.98 7.80

$Mixed.Gas
[1] 7.41 7.33 7.04

$Vacuum
[1] 5.26 5.44 5.80
```

```
> (steak.means <- as.numeric(lapply(steak.groups, mean)))
```

```
[1] 3.36 7.48 7.26 5.50
```

```
>
```

The usual parameters

We define the typical parameters for a completely randomized design.

```
> (t <- length(levels(steak.data$Treatment)))
```

```
[1] 4
```

```
> (N <- length(steak.data$LogCount))
```

```
[1] 12
```

```
> (r <- N / t)
```

```
[1] 3
```

```
>
```

Degrees of freedom for Sheffe's Test

```
> (df1 <- t - 1)
```

```
[1] 3
```

```
> (df2 = N - t)
```

```
[1] 8
```

```
>
```

Calculating the projection vector

The following two commands arrange for the groups means to be in the correct order. We note that `steak.means` contains the group means in the sorted order of the factor levels that R internally computes. Thus, we first convert the factor to a numeric vector, which results in vector `idx` of positions within the array of factor levels. Then we apply `steak.means[idx]` to list the means in the correct order.

```
> (idx <- as.numeric(steak.data$Treatment))
```

```
 [1] 2 2 2 4 4 4 3 3 3 1 1 1
```

```
> (steak.proj <- steak.means[idx])
```

```
 [1] 7.48 7.48 7.48 5.50 5.50 5.50 7.26 7.26 7.26 3.36 3.36 3.36
```

```
>
```

Estimating standard error

```
> (SSE <- sum((steak.data$LogCount - steak.proj)^2))
```

```
[1] 0.9268
```

```
> (MSE <- SSE / (N - t))
```

```
[1] 0.11585
```

```
> (steak.se.estimator <- sqrt(MSE))
```

```
[1] 0.3403674
```

```
>
```

Calculating the contrast estimator

```
> (steak.contrast.estimator <-
  as.numeric(crossprod(steak.contrast, steak.means)))
```

```
[1] 10.16
```

```
> (steak.contrast.norm <- sqrt(sum(steak.contrast^2 / r)))
```

```
[1] 2
```

Studentizing the contrast estimator

```
> (steak.contrast.se.estimator <- steak.se.estimator *
  steak.contrast.norm)
```

```
[1] 0.6807349
```

```
>
```

Calculating the Sheffe statistic

The first step is just to calculate the studentized value of the contrast.

```
> (sheffe.statistic <- steak.contrast.estimator /
  steak.contrast.se.estimator)
```

```
[1] 14.92505
```

```
>
```

Finding the critical level

```
> (alpha.E <- 0.05)
```

```
[1] 0.05
```

```
> (F.value <- pf(alpha.E, lower.tail = F, df1 = df1, df2 = df2))
```

```
[1] 0.9841499
```

```
> (sheffe.critical.level <- sqrt((t - 1) * F.value))
```

```
[1] 1.718269
```

```
> abs(sheffe.statistic) > sheffe.critical.level
```

```
[1] TRUE
```

```
>
```

Thus, we reject the null hypothesis. Let us calculate the significance level corresponding to the value of the statistic:

```
> pf(sheffe.statistic^2 / (t - 1), df1 = df1, df2 = df2,
  lower.tail = F)
```

```
[1] 3.50535e-06
```

```
>
```

Comparison with the Student t-test

If our contrast were a planned contrast, i.e. if we had known in advance that we will test its equality to 0, we could use the Student t-test. The calculations are as follows:

```
> t.statistic <- sheffe.statistic
```

```
> (t.critical.level <- pt(alpha.E / 2, df = df2, lower.tail = F))
```

```
[1] 0.4903337
```

```
> pt(abs(t.statistic), df = df2, lower.tail = F)
```

```
[1] 2.003003e-07
```

```
>
```

As we can see, the confidence level resulting from ths t-test is smaller by an order of magnitude. It is clear that there are situations when H_0 is rejected by the t-test, and not rejected by the Sheffe's test. Careful theoretical analysis would show that the opposite cannot happen.