Inner Product Spaces and Related Notions

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Inner product spaces

Definition 1. *An inner product space is a pair* $(V, \langle\cdot,\cdot\rangle)$ *where* $\langle\cdot,\cdot\rangle: V \times V \to \mathbb{R}$ *be an inner product (a bilinear, symmetric, positive definite form). Concisely formulated, the conditions on the inner product are: for all* $u, v, w \in V$ *and* $\alpha, \beta \in \mathbb{R}$*:*

1. $\langle u, v\rangle = \langle v, u\rangle$ *(symmetry)*
2. $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$. *(linearity in the first argument)*
3. $\langle u, u\rangle \geq 0$ *and equality holds only when* $u = 0$. *(positivity)*

Metric spaces

Definition 2. *A metric space is a pair* (X, d)*, where* X *is a set and* $d: X \times X \to \mathbb{R}$ *is a metric, i.e. a function satisfying these conditions for all* $u, v, w \in X$*:*

1. $d(u, v) = d(v, u)$ *(symmetry)*
2. $d(u, w) \leq d(u, v) + d(v, w)$ *(triangle inequality)*
3. $d(u, v) = 0$ *iff* $u = v$.

Norms

Definition 3. *A normed vector space is a pair* $(V, \|\cdot\|)$ *where* $\|\cdot\|: V \to \mathbb{R}$ *is a norm, i.e. a function satisfying the following conditions for all* $u, v \in V$ *and* $\alpha \in R$*:*

1. $\|u+v\| \leq \|u\| + \|v\|$.
2. $\|\alpha u\| \leq |\alpha| \|v\|$
3. $\|u\| = 0$ *iff* $u = 0$.

Every normed vector space is also a metric space, if we define the metric:

$$d(u, v) = \|u - v\|$$

More precisely, the pair (V, d) is a metric space.

Norms and distances in inner-product spaces

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. It is also a normed space and a metric space. The norm and the metric are defined as follows:

$$\begin{aligned} \|u\| &= \sqrt{\langle u, v \rangle} \\ d(u, v) &= \|u - v\| \end{aligned}$$

Orthogonal sets

Definition 4. *Let* $(V, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be an inner product space and let* $A = \{e_1, e_2, \ldots, e_n\} \subseteq V$ *be a finite subset of* V*. The set* A *is called an* *orthogonal set* *if* $\langle e_i, e_j\rangle = 0$ *for* $i \neq j$*. An infinite set* $A \subseteq V$ *is called an* *orthogonal set* *if every finite subset* $A' \subseteq A$ *is an orthogonal set. Equivalently, a subset* $A \subseteq V$ *is orthogonal iff:*

$$\forall a, b \in A \colon a \neq b \Rightarrow \langle a, b\rangle = 0.$$

Lemma 5. *An orthogonal set* $A \subseteq V$ *is linearly independent iff* $0 \notin A$*.*

Proof. "$\Rightarrow$" is obvious.
"$\Leftarrow$": Proof by contradiction. Let us suppose that A is linearly dependent. Then there exists a finite subset $A' = \{e_1, e_2, \ldots, e_n\}$ which is also linearly dependent. Thus, there exists a linear combination $\sum_{i=1}^{n} \alpha_i e_i = 0$, where $\alpha_i \in \mathbb{R}$ and there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $\alpha_{i_0} \neq 0$. Let us consider the following inner product:

$$0 = \left\langle \sum_{i=1}^{n} \alpha_i e_i, e_{i_0}\right\rangle = \sum_{i=0}^{n} \alpha_i \langle e_i, e_{i_0}\rangle = \alpha_{i_0}\langle e_{i_0}, e_{i_0}\rangle.$$

Thus, $\langle e_{i_0}, e_{i_0}\rangle = 0$. By definition of inner product, this implies $e_{i0} = 0$. This contradicts the assumption $0 \notin A$.

$\square$

Corollary 6. *Let $A \subseteq V$ be an orthogonal set such that $0 \notin A$. Then:*

$$\dim V \geq \operatorname{Card} A$$

where $\operatorname{Card} A$ *denoted the cardinality (number of elements) of A.*

Definition 7. *An orthogonal set $A \subseteq V$ is called an an orthonormal set iff:*

$$\forall a \in A \colon \|a\| = 1.$$

Normalization is a process of converting an orthogonal set $A \subseteq V$ into an orthonormal set A' by normalizing every vector of a, i.e. replacing it with $a / \|a\|$. This is a trivial transformation leading to some simplification of theorems and formulas.

Orthogonal projections

Let V be an inner product space space and let Let $\{e_1, e_2, \ldots, e_n\}$ be a linearly independent, orthogonal set. Let $W = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Let $P \colon V \to V$ be a transformation given by the formula:

$$P(v) = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i.$$

Theorem 8. *The transformation P defined above has the following properties:*

1. *The transformation P is a linear transformation: for all $u, v \in V$ and for all $\alpha, \beta \in \mathbb{R}$:*
$$P(\alpha u + \beta v) = \alpha P(u) + \beta P(v).$$
2. *$P(V) = W$.*
3. *P is an idempotent: $P \circ P = P$.*
4. *P is self-adjoint: $\langle Pv, w \rangle = \langle v, Pw \rangle$.*
5. *For every $v \in V$ and $w \in W$: $v - Pv \perp w$. In short, $v - Pv \perp W$.*
6. *For every $v \in V$ the point Pv is the closest point to V which lies in W:*
$$\forall z \in W: \|v - z\| \geq \|v - Pv\|.$$
7. *For every $v \in V$: $\|Pv\| \leq \|v\|$.*

Remark 9. This theorem shows that P is the orthogonal projection onto W.

Remark 10. If $V = \mathbb{R}^n$ and $\langle \cdot\, , \cdot \rangle$ is the standard inner product, the dot product of vectors given by the formula $\langle u,\, v \rangle = u^T v$, then P is identified with a matrix:

$$P = \sum_{i=1}^{n} \frac{e_i e_i^T}{e_i^T e_i}$$

Exercise 1. Find the orthogonal projection onto the subspace of $\mathbb{R}^3$:

$$W = \operatorname{span}\{(1,1,0),(1,-1,1)\}.$$

Pythagorean Theorem

Theorem 11. *Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product space and let $A = \{e_1, e_2, ..., e_n\}$ be an orthogonal set. If $v = \sum_{i=1}^{n} \alpha_i e_i$ then:*

$$\|v\|^2 = \sum_{i=1}^{n} \alpha_i^2 \langle e_i, e_i \rangle.$$

Representation theorem

Theorem 12. *Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product space, and let $A = \{e_1, e_2, ..., e_n\}$ be an orthogonal basis. Then for every $v \in V$ we have:*

$$v = \sum_{i=1}^{n} \frac{\langle v, e_i \rangle}{\langle e_i, e_i \rangle} e_i$$

Corollary 13. *Let* $A = \{e_1, e_2, ..., e_n\}$ *be an orthogonal set. Then:*

1. *For every vector* $v \in V$*:*

$$\|v\|^2 \geq \sum_{i=1}^{n} \frac{\langle v, e_i \rangle^2}{\langle e_i, e_i \rangle}$$

2. *The above inequality is not sharp (i.e. equality holds) iff* $v \in \operatorname{span} A$*.*