A SIMULATION OF A FALLING BARBELL

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1. Barbell

A barbell is a rod of length $2\,\ell$ and mass $2\,m$. We assume it to be symmetric with respect to the center of the rod. The height of the center of mass is x. The angle the barbell forms with the horizontal direction is θ. The moment of inertia of the rod is I.

We assume that the mass is concentrated in the ends. This results in the formula for I:

$$I = 2 \times \left(m\,\ell^2 \right) = 2\,m\,\ell^2$$

The potential consists of the gravitational potential and the potential of the interaction with the floor and the ends. We assume that the potential only depends on the distance from the floor. Moreover, if

$$h = x \pm \ell\,\sin\theta$$

is the distance from the floor of one of the ends then

$$V(h) = \begin{cases} m\,g\,h & h \geq 0, \\ -\frac{1}{2}C\,h^2 & h < 0. \end{cases}$$

The first formula expresses ordinary gravitational potential. The second formula is an elastic force of the floor. We assume the floor is frictionless, which means that there is no horizontal component to the force.

The Lagrangian is

$$L = T - V$$
$$= m\dot{x}^2 + \frac{1}{2}\,I\dot{\theta}^2$$
$$- V(x + \ell\sin\theta) - V(x - \ell\sin\theta).$$

The generalized momentum is:

$$p_x = \frac{\partial L}{\partial \dot{x}} = 2m\dot{x}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

The generalized forces are:

$$F_x = \frac{\partial L}{\partial x} = -V'(x + \ell\sin\theta) - V'(x - \ell\sin\theta)$$
$$F_\theta = \frac{\partial L}{\partial \theta} = -V'(x + \ell\sin\theta)\,\ell\cos\theta + V'(x - \ell\sin\theta)\,\ell\cos\theta.$$

Let $f(h) = V'(h)$. We have

$$f(h) = \begin{cases} m\,g & h \geq 0 \\ -C\,h & h < 0 \end{cases}$$

Thus, the force is a step function.

Euler-Lagrange equations for this system are:

$$\frac{dp_x}{dt} = F_x = -\left(f(x + \ell \sin\theta) + f(x - \ell \sin\theta) \right),$$

$$\frac{dp_\theta}{dt} = F_\theta = -\left(f(x + \ell \sin\theta) - f(x - \ell \sin\theta) \right) \ell \cos\theta,$$

$$\frac{dx}{dt} = \frac{p_x}{2m},$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{I}.$$