Math 125 Notes on Multiple Proportionality

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Multiple proportionality

Example 1. Illumination I is proportional to $\cos\theta$ and $\frac{1}{r^2}$. Show that the formula for I is:

$$I = C\,\frac{\cos\theta}{r^2} \tag{1}$$

where C is a certain constant.

It is more generally true that if a quantity Q turns out to be proportional to two different functions $f(x)$ and $g(y)$ then $Q = k\,f(x)\,g(y)$ where k does not depend on x and y.

Remark 2. Note that such multiple proportionality is determined by two different experiments. For instance, one varies the angle θ first, while keeping r fixed, for several values of r. Subsequently, one varies r while keeping θ fixed, and one conducts the experiment with several values of θ.

Proof. We can write $Q = C_1(y)\,f(x)$ and $Q = C_2(x)\,g(y)$. Thus

$$C_1(y)\,f(x) = C_2(x)\,g(y)$$

$$\frac{f(x)}{C_2(x)} = \frac{g(y)}{C_1(y)}$$

We observe that the left-hand side is a function of x only, and the right-hand side is a function of y only. The only possibility is that both functions are constant. Let us call the constant C. Thus:

$$C = \frac{f(x)}{C_2(x)} = \frac{g(y)}{C_1(y)}$$

Hence, $C_1(y) = g(y)/C$ and $C_2(x) = f(x)/C$. In particular,

$$Q = C_1(y)\, f(x) = \frac{g(y)}{C}\, f(x) = \frac{1}{C}\, f(x)\, g(y)$$

Hence, $Q = k\, f(x)\, g(y)$ where $k = 1/C$.

$\square$